Embedded Systems
2012/13

Basilica di Santa Maria di Collemaggio, 1287, L'Aquila

Lecture 8
Symbolic Control Design

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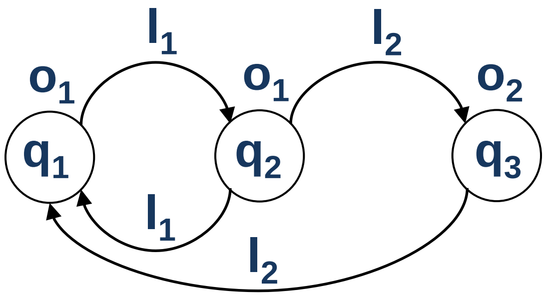
A unified framework for continuous and discrete systems

Definition A transition system is a tuple:

$$T = (Q, Q_0, L, \longrightarrow, O, H),$$

consisting of:

- a set of states Q
- a set of initial states Q_0
- a set of labels L
- a transition relation $\longrightarrow \subseteq Q \times L \times Q$
- an output set O
- an output function $H: Q \to O$



We will follow standard practice and denote $(q, l, q') \in \longrightarrow$ by $q \xrightarrow{l} q'$

Review: preliminary definitions

A transition system T is said to be:

- countable if Q and L are countable sets
- symbolic/finite if Q and L are finite sets
- non-blocking if for $q \in Q$ there exists $l \in L$ and $q' \in Q$ such that $q \xrightarrow{l} q'$
- deterministic if for any $q \in Q$ and any $l \in L$ there exists at most one state $q' \in Q$ s.t. $q \xrightarrow{l} q'$
- accessible if for any $q \in Q$ there exists an initial state $q_0 \in Q_0$ and a finite path

$$q_0 \xrightarrow{l_1} q_1 \xrightarrow{l_2} \ldots\ q_{n-1} \xrightarrow{l_n} q_n$$

ending up in $q_n = q$.

- A transition system $T_1 = (Q_1, Q_{01}, L_1, \longrightarrow_1, O_1, H_1)$ is a sub-transition system of a transition system $T_2 = (Q_2, Q_{02}, L_2, \longrightarrow_2, O_2, H_2)$ if $Q_1 \subseteq Q_2$, $Q_{01} \subseteq Q_{02}$, $L_1 \subseteq L_2$, $\longrightarrow_1 \subseteq \longrightarrow_2$, $O_1 \subseteq O_2$, and $H_1(q)=H_2(q)$ for all states $q \in Q_1$. We write $T_1 \sqsubseteq T_2$.

- The non-blocking part $Nb(T)$ of a transition system T is the unique non-blocking transition system s.t. $T' \sqsubseteq Nb(T) \sqsubseteq T$ for any non-blocking transition system T'.

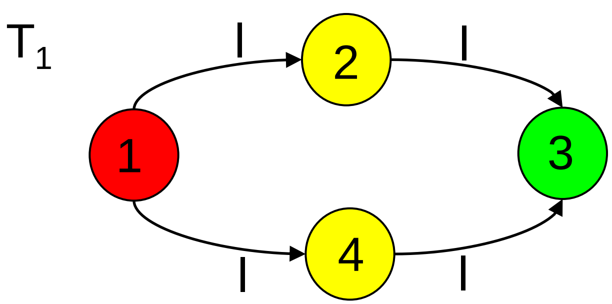
Review: bisimulation equivalence

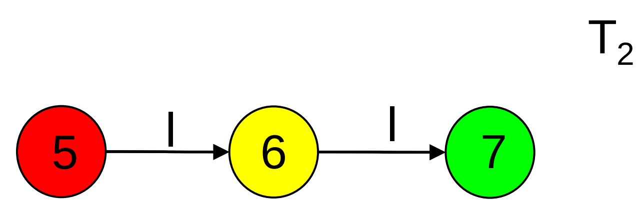
LTSs $T_1 = (Q_1, Q_{01}, L_1, \longrightarrow_1, O_1, H_1)$ and $T_2 = (Q_2, Q_{02}, L_2, \longrightarrow_2, O_2, H_2)$, with $O_1 = O_2$, are **bisimilar** if there exists a relation

$$R \subseteq Q_1 \times Q_2$$

(called **bisimulation relation** between T_1 and T_2) such that:

- For any $q_1 \in Q_{01}$ there exists $q_2 \in Q_{02}$ such that $(q_1, q_2) \in R$
- For any $q_2 \in Q_{02}$ there exists $q_1 \in Q_{01}$ such that $(q_1, q_2) \in R$
- For any $(q_1, q_2) \in R$, $H_1(q_1) = H_2(q_2)$
- For any $(q_1, q_2) \in R$, $q_1 \xrightarrow{l_1}_1 p_1$ in T_1 implies existence of $q_2 \xrightarrow{l_2}_2 p_2$ in T_2 s.t. $(p_1, p_2) \in R$
- For any $(q_1, q_2) \in R$, $q_2 \xrightarrow{l_2}_2 p_2$ in T_2 implies existence of $q_1 \xrightarrow{l_1}_1 p_1$ in T_1 s.t. $(p_1, p_2) \in R$





R = { (1,5), (2,6), (3,7), (4,6) }

Review: approximate bisimulation

LTSs $T_1 = (Q_1, Q_{01}, L_1, \longrightarrow_1, O_1, H_1)$ and $T_2 = (Q_2, Q_{02}, L_2, \longrightarrow_2, O_2, H_2)$, with $O_1 = O_2$, are **ε - bisimilar** for some precision $\varepsilon > 0$, if there exists a relation

$$R \subseteq Q_1 \times Q_2$$

(called **approximate bisimulation relation** between T_1 and T_2) such that:

- For any $q_1 \in Q_{01}$ there exists $q_2 \in Q_{02}$ such that $(q_1, q_2) \in R$
- For any $q_2 \in Q_{02}$ there exists $q_1 \in Q_{01}$ such that $(q_1, q_2) \in R$
- For any $(q_1, q_2) \in R$, $H_1\cancel{(q_1) = H_2(q_2)}$ $d(H_1(q_1), H_2(q_2)) \le \varepsilon$
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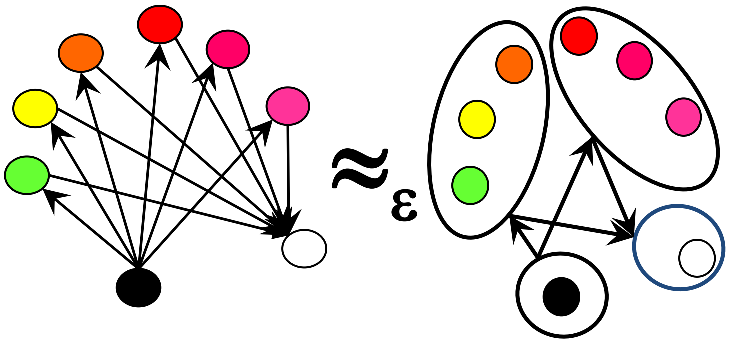
An **approximate simulation relation** is a one-sided version of an **approximate** bisimulation relation



Review: Transition Systems modeling Control Systems

A **nonlinear control system** Σ

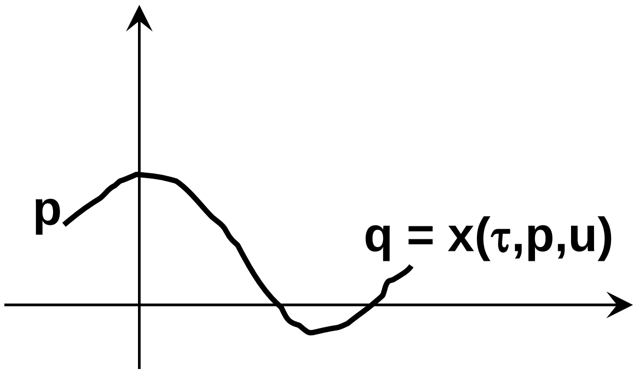
$$dx/dt = f(x,u),\ x \in X \subseteq R^n,\ u \in U \subseteq R^m$$

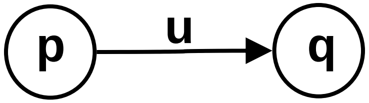
can be modeled by the transition system

$$T(\Sigma) = (X, X_0, \mathcal{U}, \longrightarrow, O, H),$$

where:

- $X_0 = X$ is a set of initial states
- $\mathcal{U}$ is the collection of control signals $u : R \rightarrow U$
- $p \xrightarrow{u} q$, if $x(\tau,p,u) = q$ for some $\tau \geq 0$
- $O = X$
- H is the identity function





$T(\Sigma)$ captures information contained in Σ but it is not a symbolic model because X and U are infinite sets!

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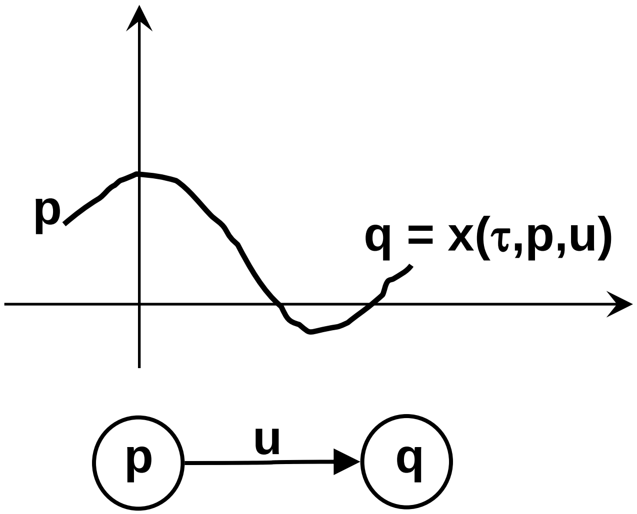

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Software can be modelled by systems

The states are all the possible memory configurations and the transition relation describes how the memory contents are changed by the execution of instructions

Review: construction of symbolic models

We consider digital control systems, i.e. control systems where input signals are piecewise constant.

Consider a nonlinear digital control system

$$T(\Sigma) = (X, X_0, \mathcal{U}, \longrightarrow, O, H),$$

and given some $\tau > 0$, define the transition system

$$T_\tau(\Sigma) = (X, X_0, \mathcal{U}_\tau, \longrightarrow_\tau, O, H),$$

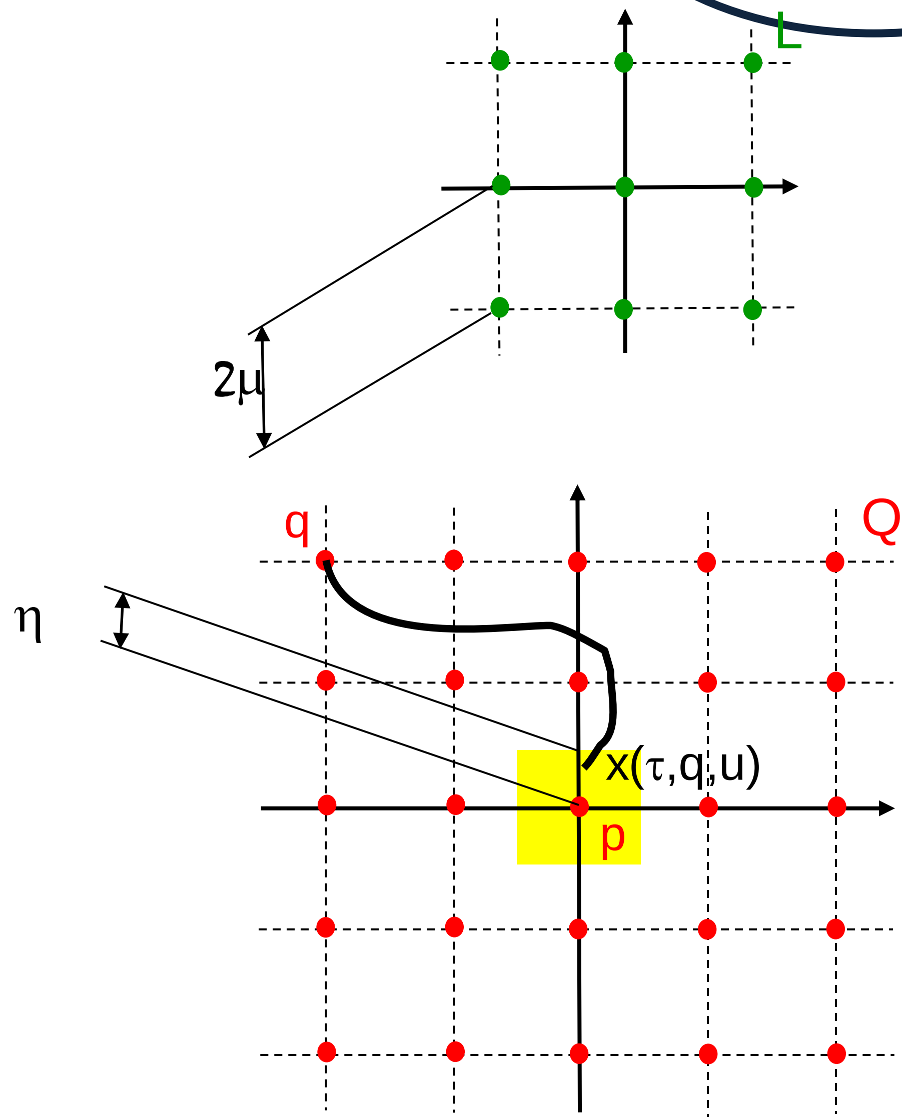
where:

- $\mathcal{U}_\tau$ is the collection of constant functions $u : [0,\tau] \rightarrow R^m$
- $p \xrightarrow{u}_\tau q$ if $x(\tau,p,u) = q$

Review: construction of symbolic models

Consider the following parameters:

- $\tau > 0$ sampling time
- $\eta > 0$ state space quantization
- $\mu > 0$ input space quantization



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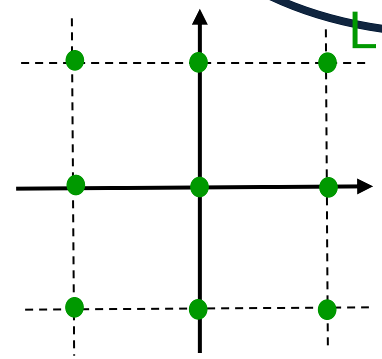

- $\tau > 0$ sampling time
- $\eta > 0$ state space quantization
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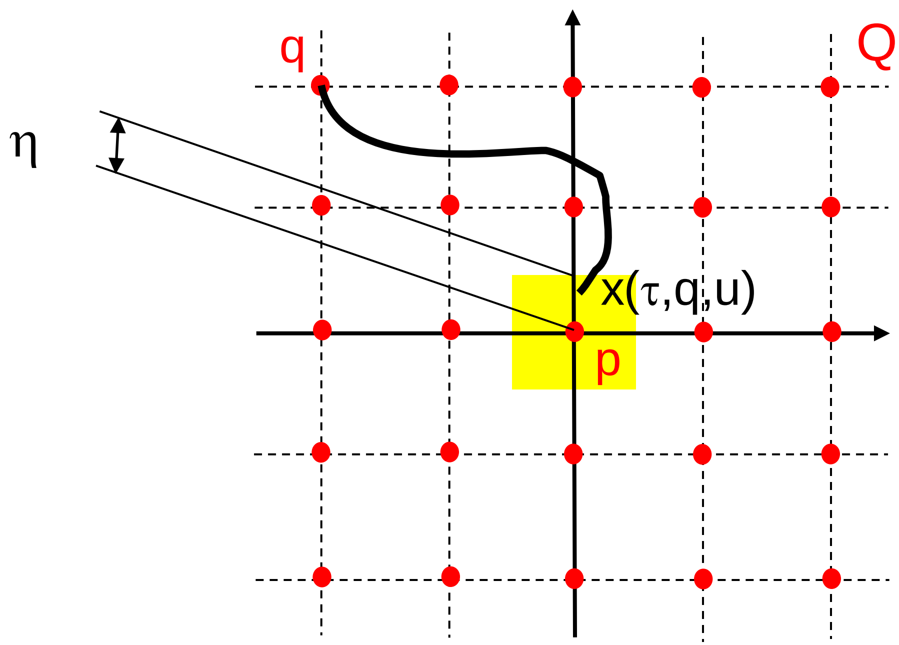
and define $T_{\tau,\eta,\mu}(\Sigma) = (X_{\tau,\eta,\mu}, X_{0,\tau,\eta,\mu}, U_{\tau,\eta,\mu}, \longrightarrow_{\tau,\eta,\mu}, O, H)$, where:

- $X_{\tau,\eta,\mu} = [X]_{2\eta}$
- $X_{0,\tau,\eta,\mu} = X_{\tau,\eta,\mu} \cap X_0$
- $U_{\tau,\eta,\mu} = [U]_{2\mu}$
- $p \xrightarrow{u}_{\tau,\eta,\mu} q$, if $\|x(\tau,p,u) - q\| \leq \eta$
- $O = X$
- H is the identity function



Remark

Transition system $T_{\tau,\eta,\mu}(\Sigma)$ is countable.
If state and input spaces of Σ are bounded then $T_{\tau,\eta,\mu}(\Sigma)$ is symbolic!

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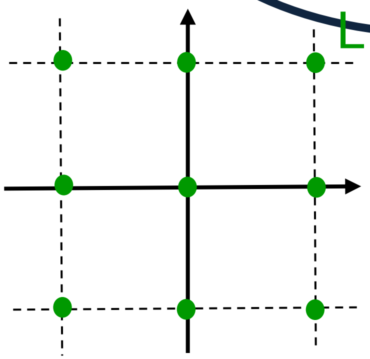

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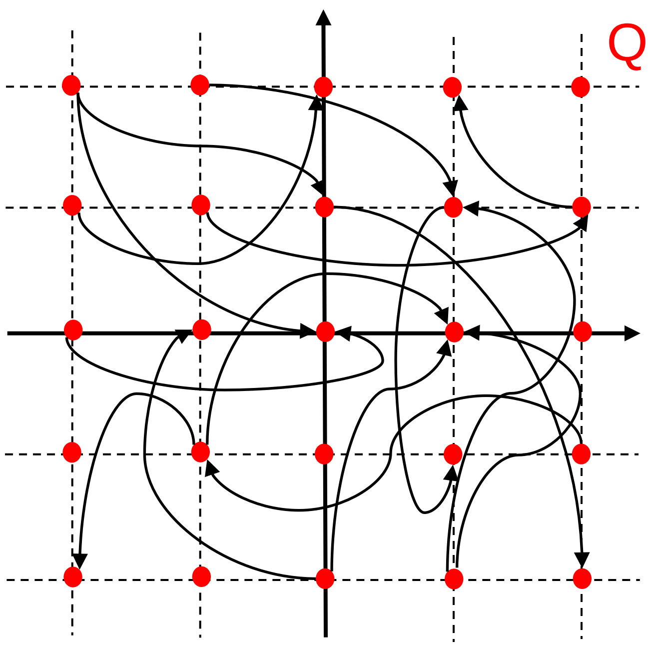

Theorem If Σ is δ-ISS, for any desired precision $\varepsilon > 0$ and for any $\tau, \eta, \mu > 0$ satisfying

$$\beta(\varepsilon,\tau) + \eta + \gamma(\mu) \leq \varepsilon$$

then $T_\tau(\Sigma)$ and $T_{\tau,\eta,\mu}(\Sigma)$ are ε-bisimilar

Design of symbolic controllers

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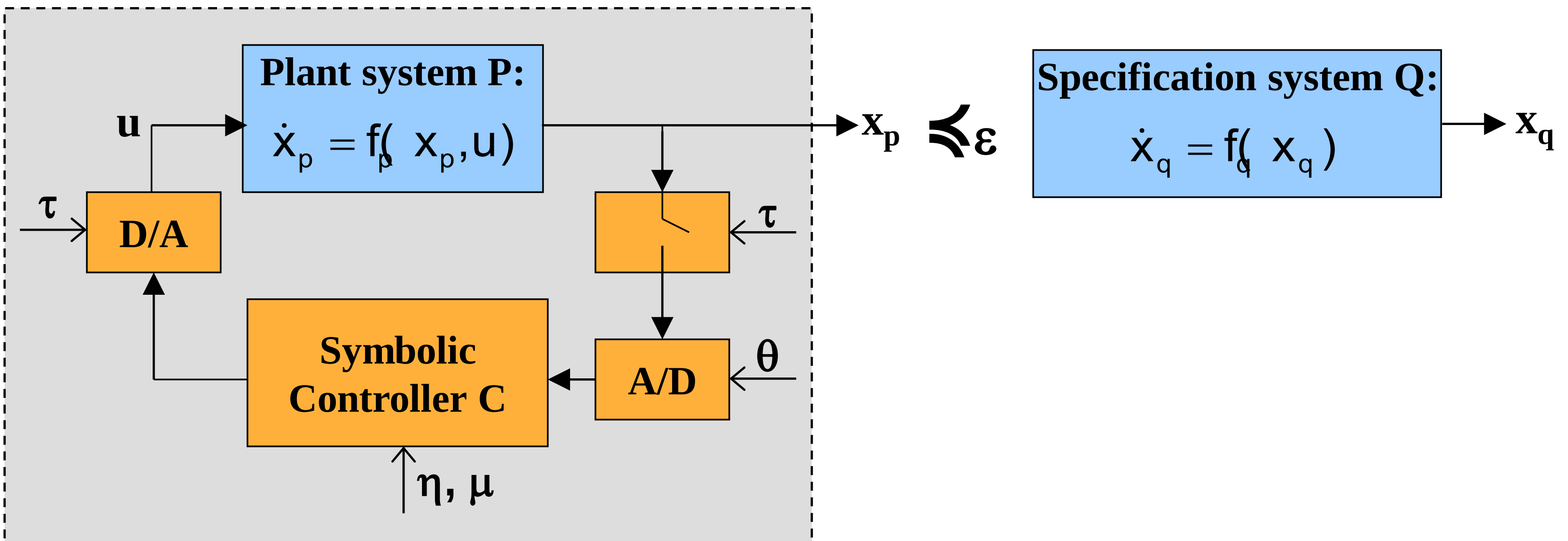
Problem 1: Continuous Specifications [cf. Borri, Pola, Di Benedetto, CDC 2010]

Given a plant P, a specification Q and a desired precision $\varepsilon > 0$, find a symbolic controller that implements Q up to the precision ε and that is non-blocking when interacting with P.



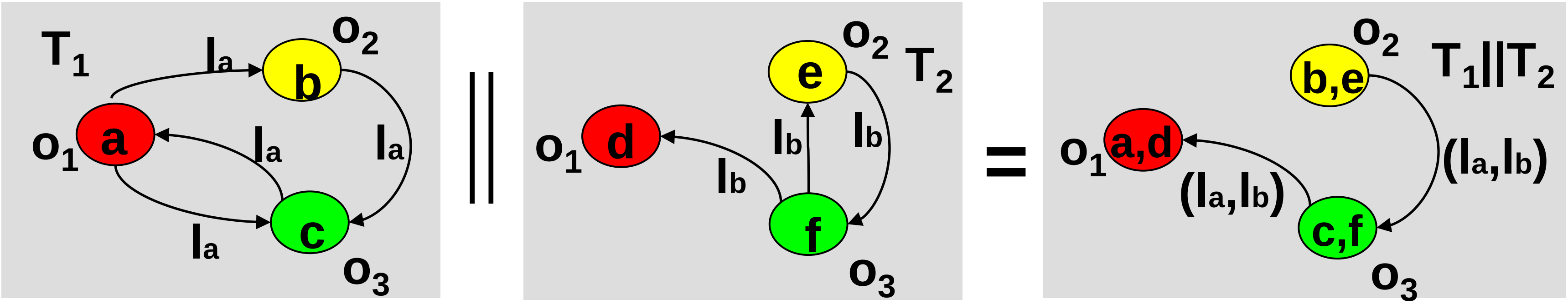
Approximate composition [Tabuada IEEE TAC 08]

Definition Given $T_1 = (Q_1, Q_{01}, L_1, \longrightarrow_1, O_1, H_1)$ and $T_2 = (Q_2, Q_{02}, L_2, \longrightarrow_2, O_2, H_2)$, with $O_1 = O_2$, and a precision $\theta > 0$, the approximate composition of T_1 and T_2 is the system

$$T = T_1 \,||_\theta\, T_2 = (Q, Q_0, L, \longrightarrow, O, H)$$

where:

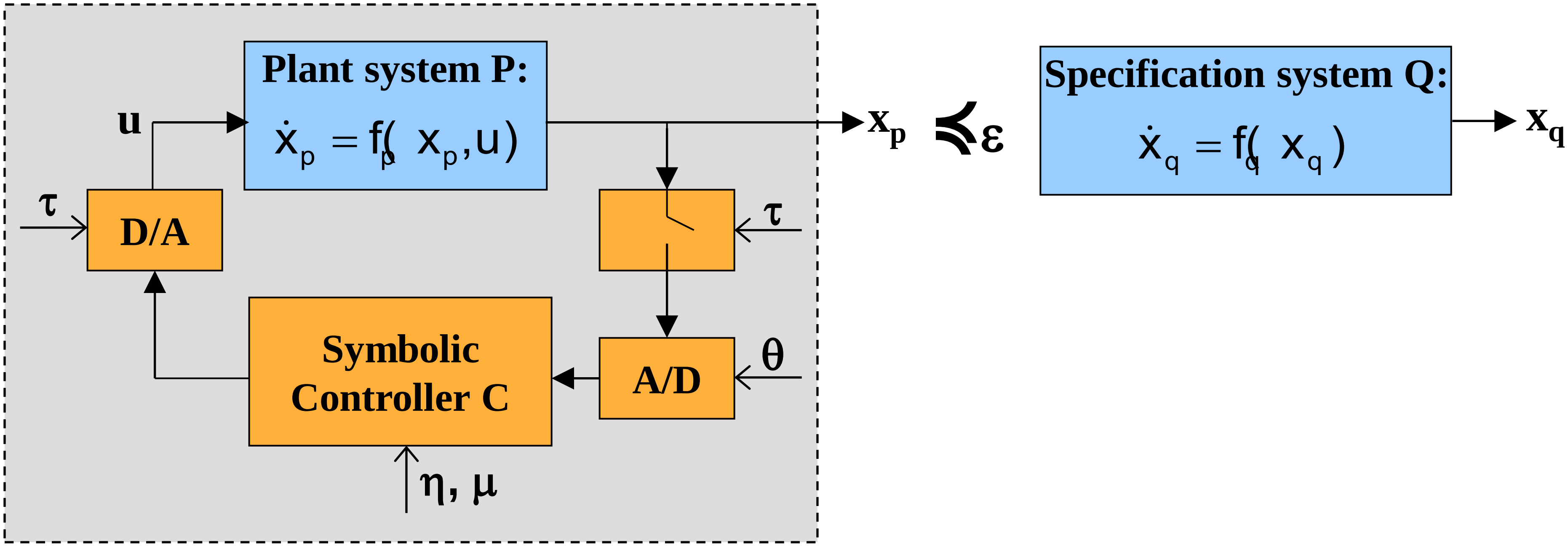
- $Q = \{(q_1, q_2) \in Q_1 \times Q_2 : d(H_1(q_1), H_2(q_2)) \leq \theta\}$
- $Q_0 = Q \cap (Q_{01} \times Q_{02})$
- $L = L_1 \times L_2$
- $(q_1,q_2) \xrightarrow{(l_1,l_2)} (p_1,p_2)$, if $q_1 \xrightarrow{l_1} p_1$ and $q_2 \xrightarrow{l_2} p_2$
- $O = O_1 = O_2$
- $H(q_1,q_2) = H_1(q_1)$



Design of symbolic controllers

Problem 1
Given a plant P, a specification Q and a desired precision $\varepsilon > 0$, find a symbolic controller C such that

1. $T_\tau(P)||_\theta C \preccurlyeq_\varepsilon T_\tau(Q)$
2. $T_\tau(P)||_\theta C$ is non-blocking



Solution of Problem 1

Synthesis through a four-step process:

1. Compute the symbolic model $T_{\tau,\eta,\mu}(P)$ of P
2. Compute the symbolic model $T_{\tau,\eta,0}(Q)$ of Q
3. Compute the symbolic controller $C^* = T_{\tau,\eta,\mu}(P) \| T_{\tau,\eta,0}(Q)$
4. Compute the non-blocking part Nb(C*) of C*

Plant P:
Continuous System

Specification Q:
Continuous System

Solution of Problem 1

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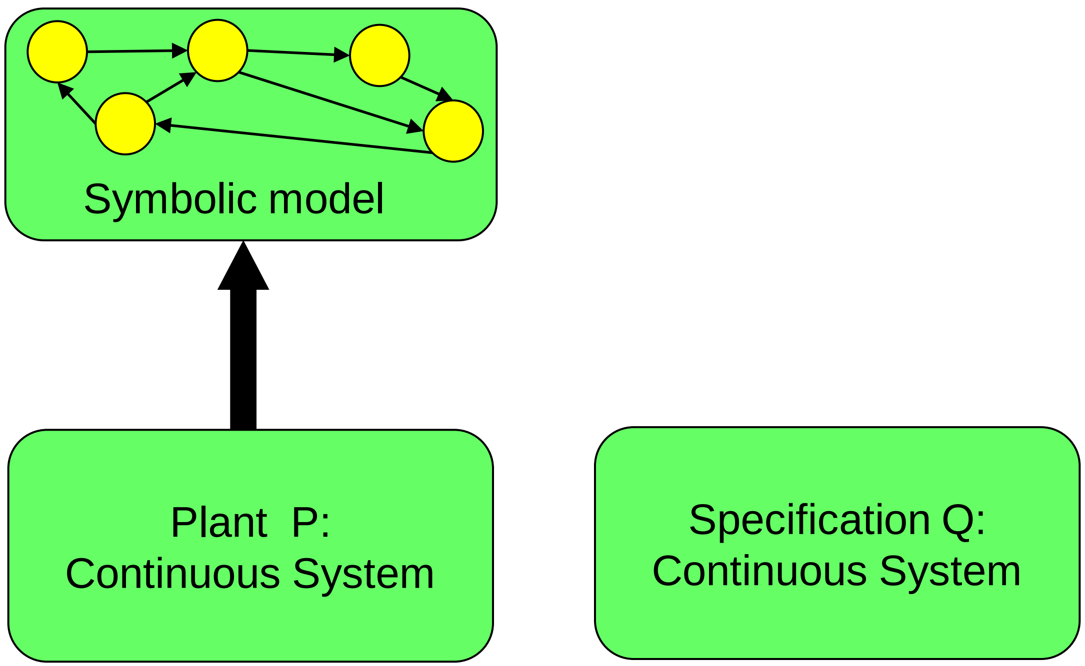

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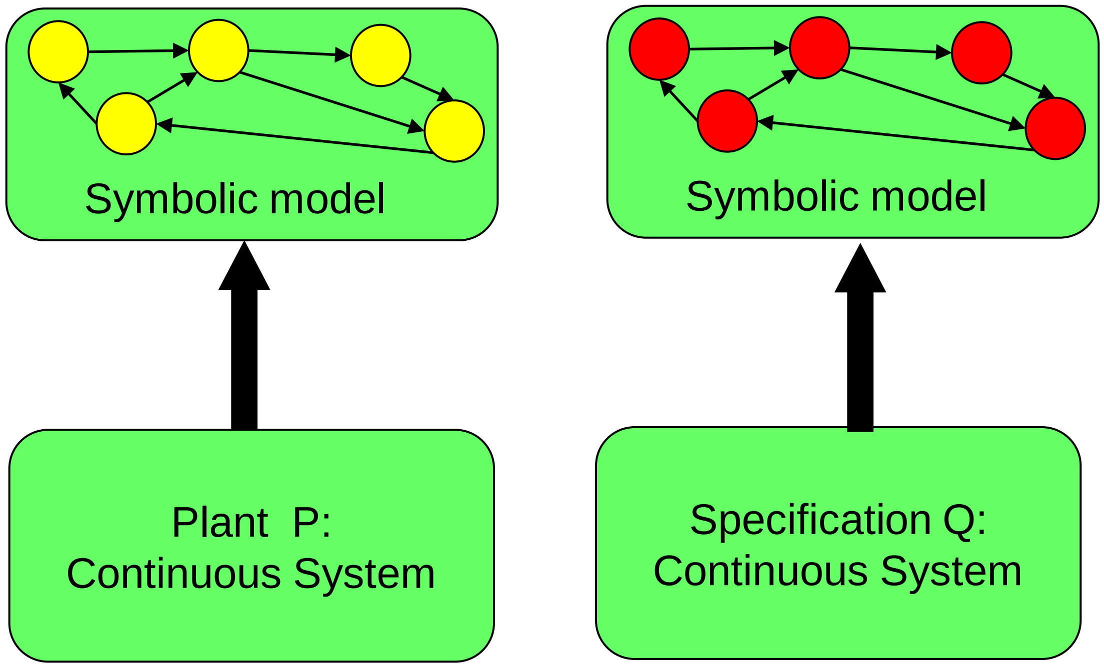

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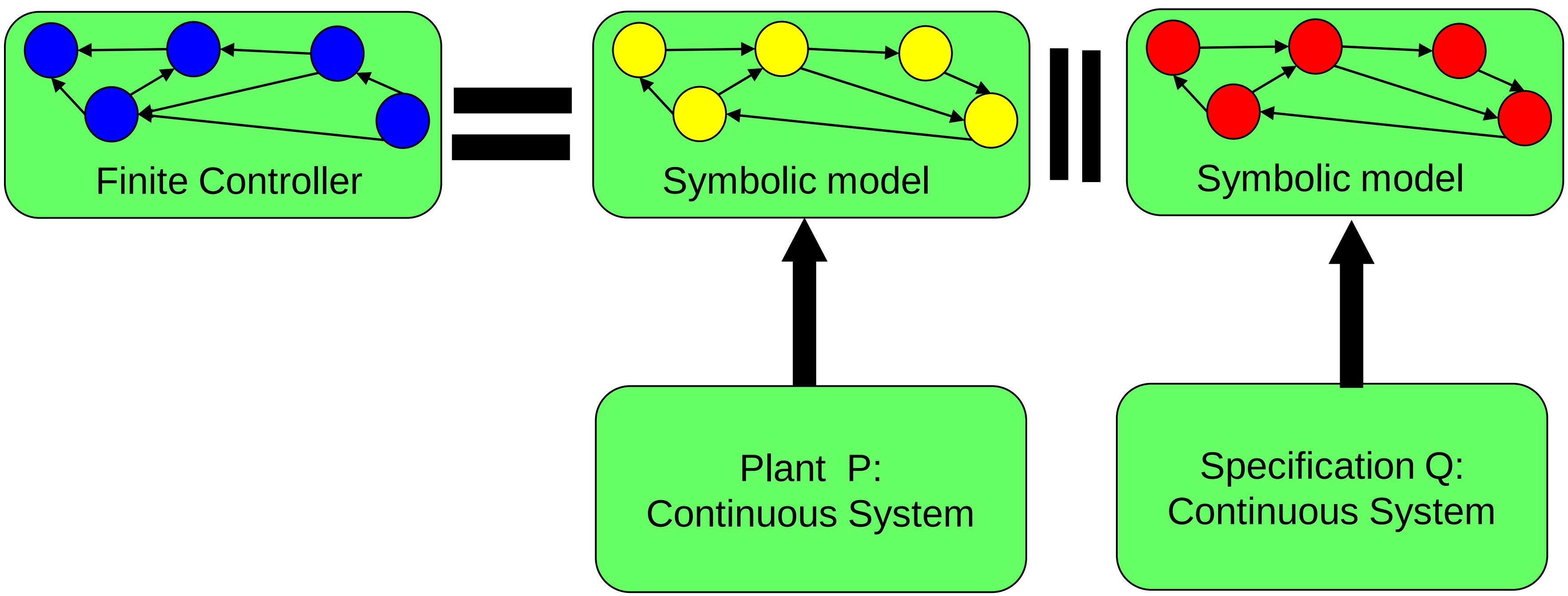

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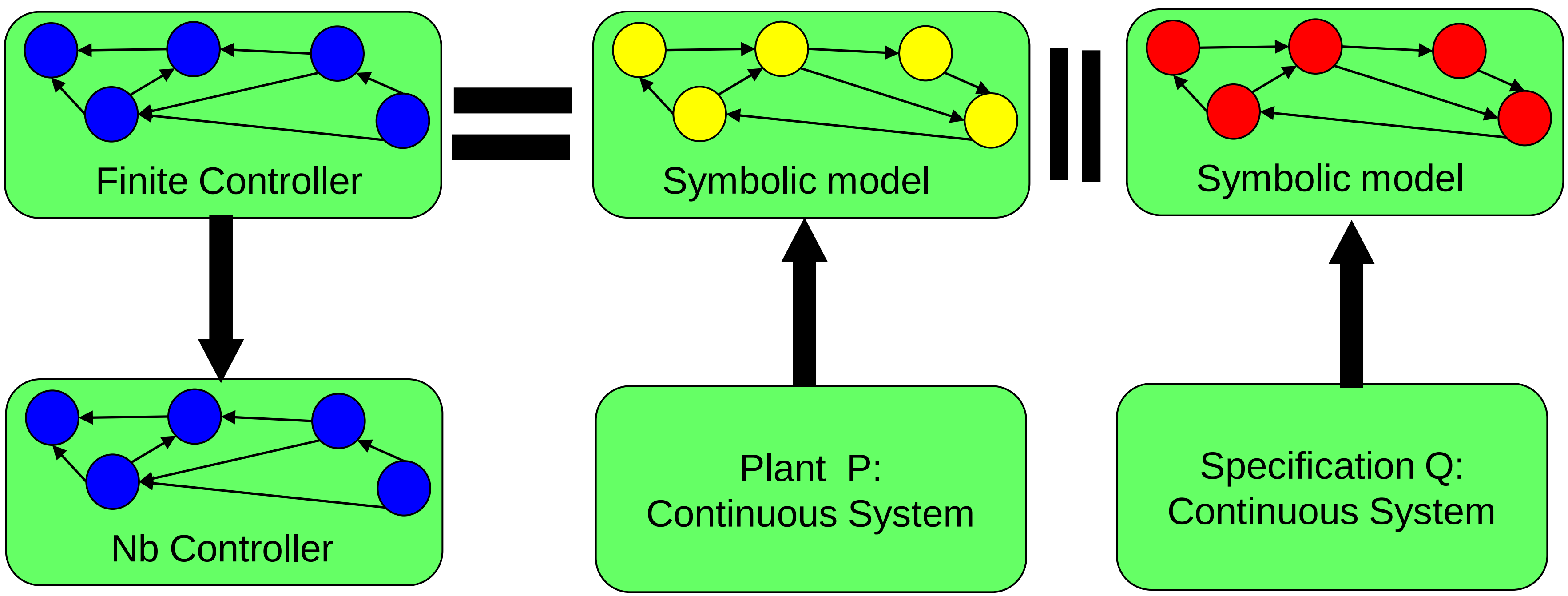

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3. Compute the symbolic controller $C^* = T_{\tau,\eta,\mu}(P) || T_{\tau,\eta,0}(Q)$
4. Compute the non-blocking part $Nb(C^*)$ of C^*

Theorem Suppose that P and Q are δ–ISS and choose parameters $\varepsilon_p, \varepsilon_q > 0$ so that

(1) $$\varepsilon_p + \varepsilon_q \leq \varepsilon$$

Choose parameters $\tau, \eta, \mu > 0$ satisfying

(2) $$\beta_p(\varepsilon_p,\tau) + \eta + \gamma_p(\mu) \leq \varepsilon_p$$

(3) $$\beta_q(\varepsilon_q,\tau) + \eta \leq \varepsilon_q$$

The symbolic controller $Nb(C^*)$ solves Problem 1 with $\theta = \varepsilon_p$

Design of symbolic controllers

Drawbacks

- It considers the whole sets of states of $T_{\tau,\eta,\mu}(P)$ and $T_{\tau,\eta,0}(Q)$
- For any source state x and target state y, it includes all transitions $x \xrightarrow{u} y$ with any control input u by which state x reaches state y
- It first constructs $T_{\tau,\eta,\mu}(P)$ and $T_{\tau,\eta,0}(Q)$, then C^*, to finally eliminate blocking states from C^*

To cope with space and time complexity, instead of computing separately

(1) Discrete abstraction $T_{\tau,\eta,\mu}(P)$ of P
(2) Discrete abstraction $T_{\tau,\eta,0}(Q)$ of Q
(3) Symbolic controller $C^* = T_{\tau,\eta,\mu}(P) || T_{\tau,\eta,0}(Q)$
(4) Non-blocking part $Nb(C)$ of C^*

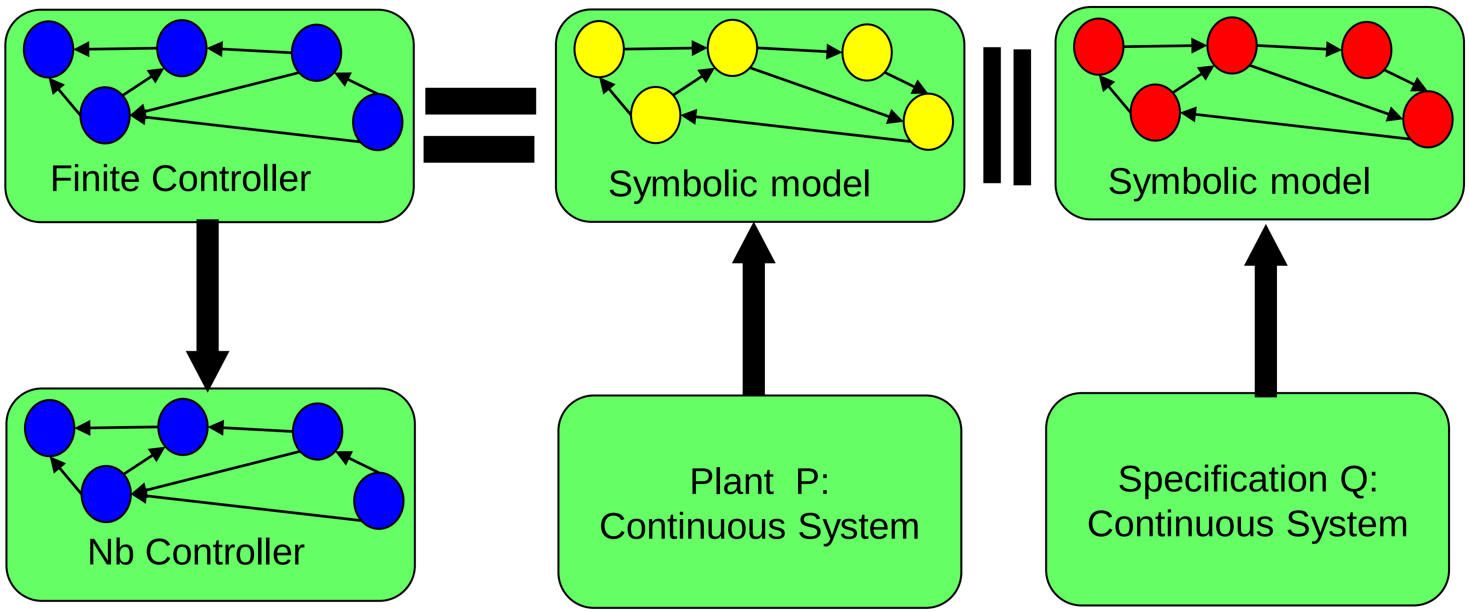
Integrated Approach: Compute (1) + (2) + (3) + (4) at once!

Space/time complexity analysis of the proposed algorithm formally quantifies the gain of the integrated approach

Integrated Algorithm for Problem 1

Basic ideas

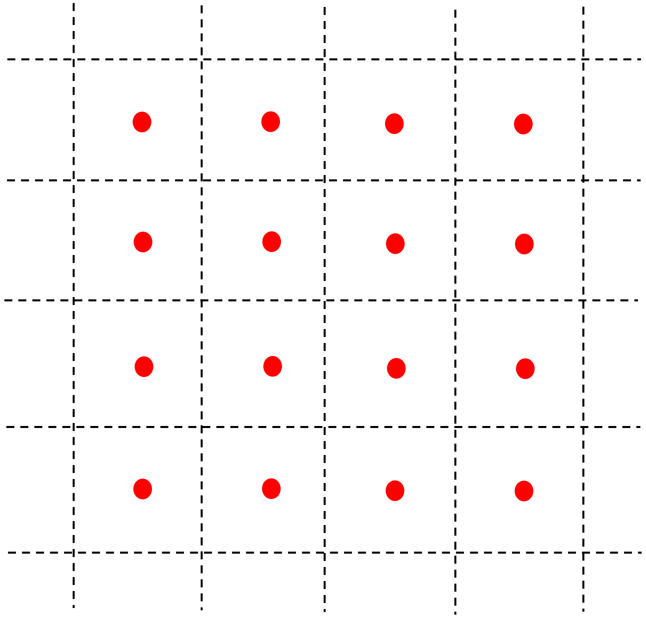
1. It only considers the intersection of the accessible parts of P and Q
2. For any given source state x and target state y, it considers only one transition (x,u,y)
3. It eliminates blocking states as soon as show up



Integrated Algorithm for Problem 1

How does it work?

First, we consider the target space as the intersection of the sets of initial states of $T_{\tau,\eta,\mu}(P)$ and $T_{\tau,\eta,0}(Q)$.



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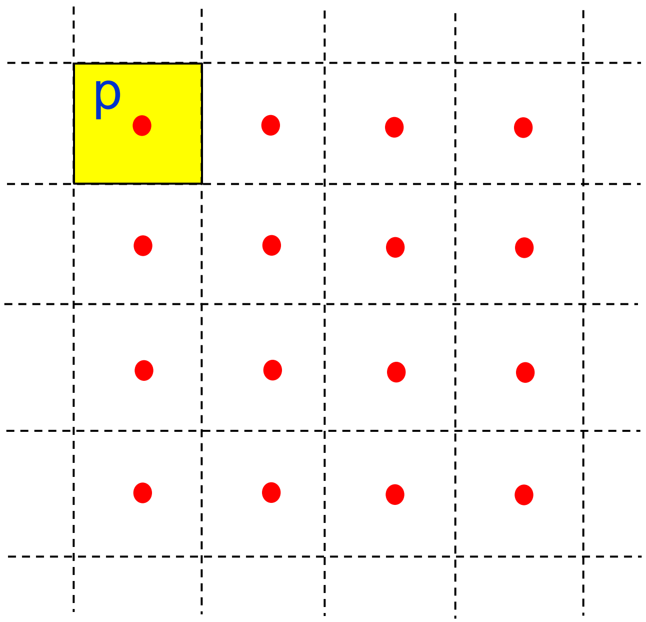
First, we consider the target space as the intersection of the sets of initial states of $T_{\tau,\eta,\mu}(P)$ and $T_{\tau,\eta,0}(Q)$.

Pick a "symbolic" state p from the target space and compute $[x_q(\tau,p)]_{2\eta}=q$ by integrating the specification differential equation



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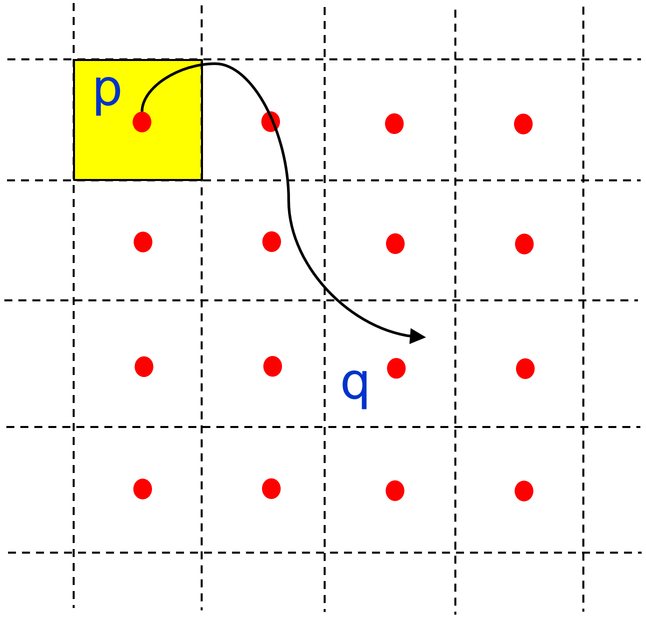

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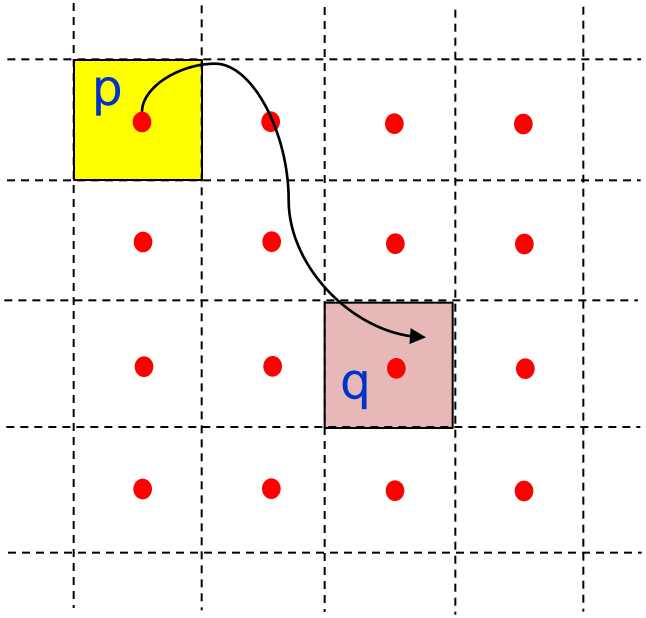

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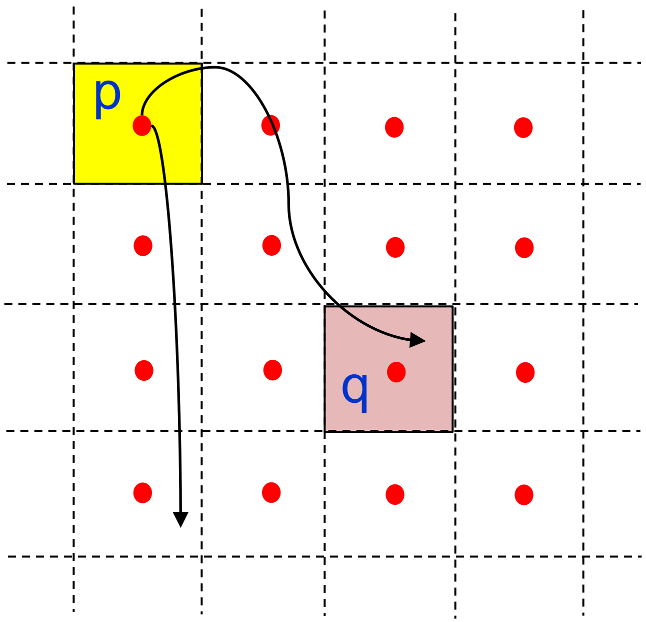
Pick control inputs in $[U]_{2\mu}$ and integrate the plant differential equation until $q=[x_p(\tau,p,u)]_{2\eta}$ for some u.



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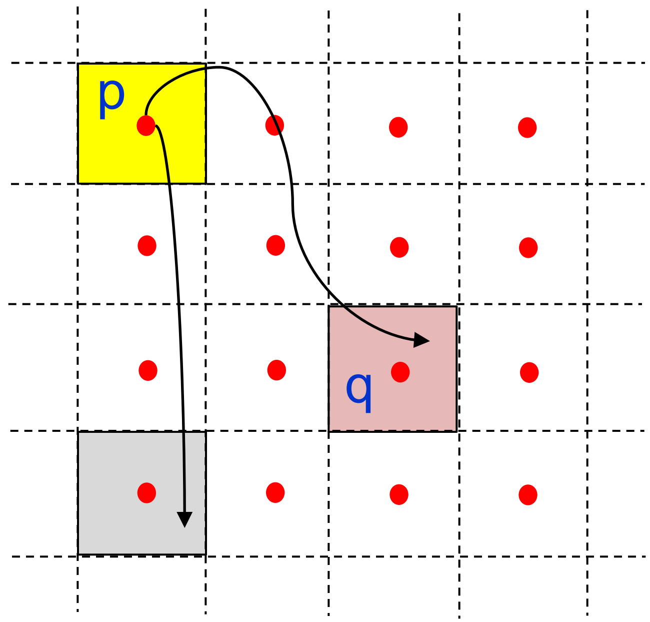

No matching! Try another input!

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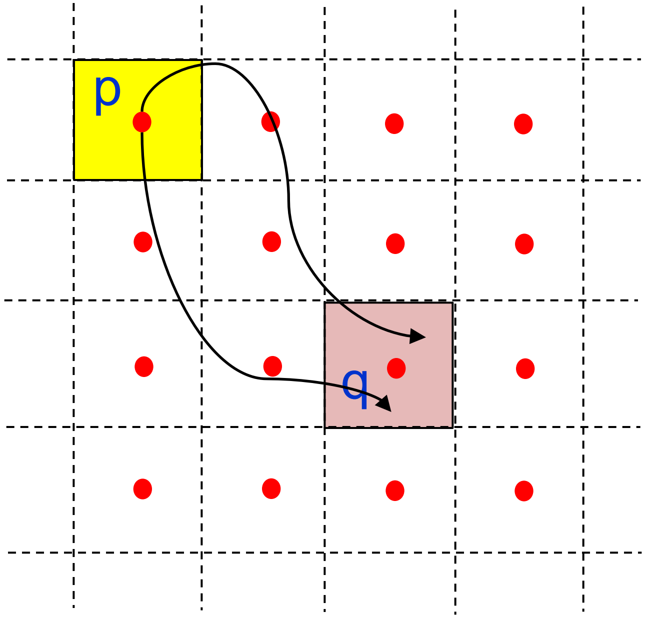

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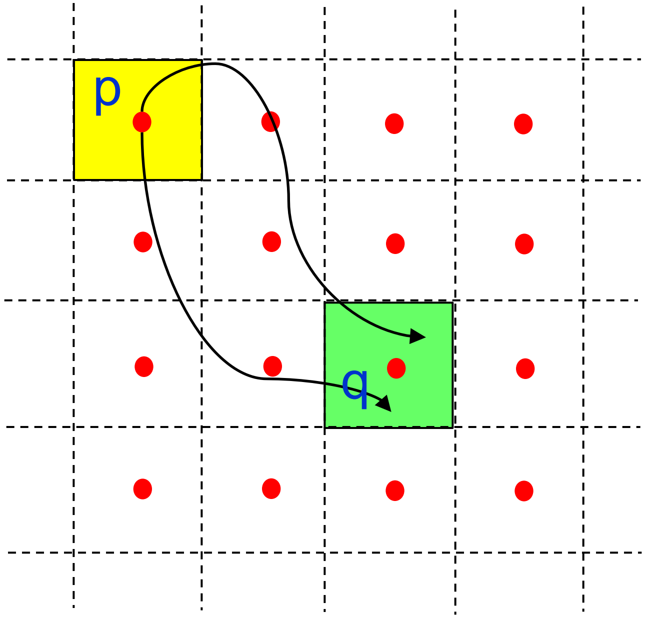

Matching found!!

Integrated Algorithm for Problem 1

DEWS CENTER OF EXCELLENCE

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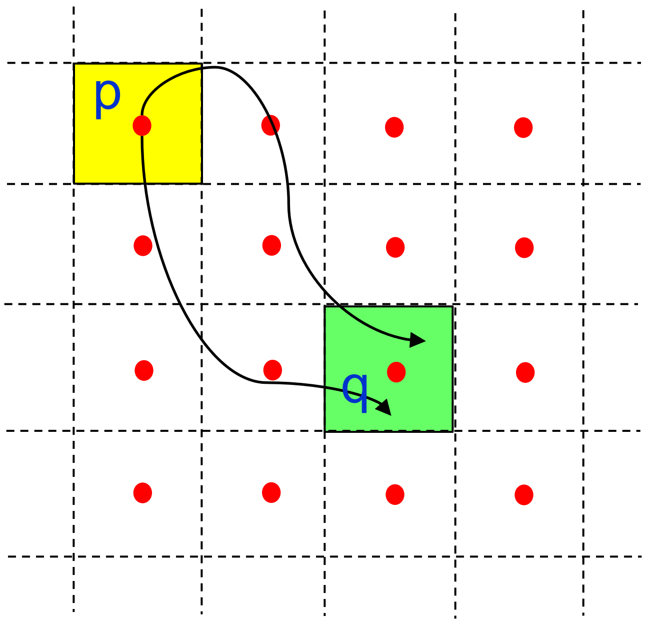

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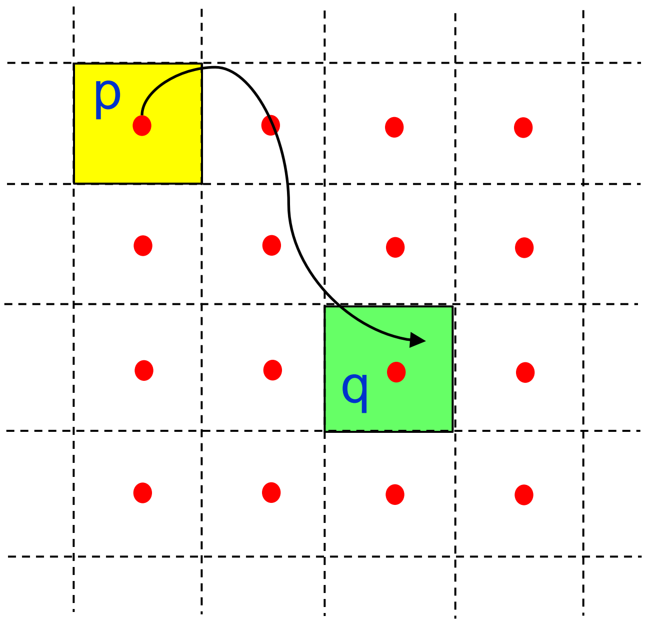

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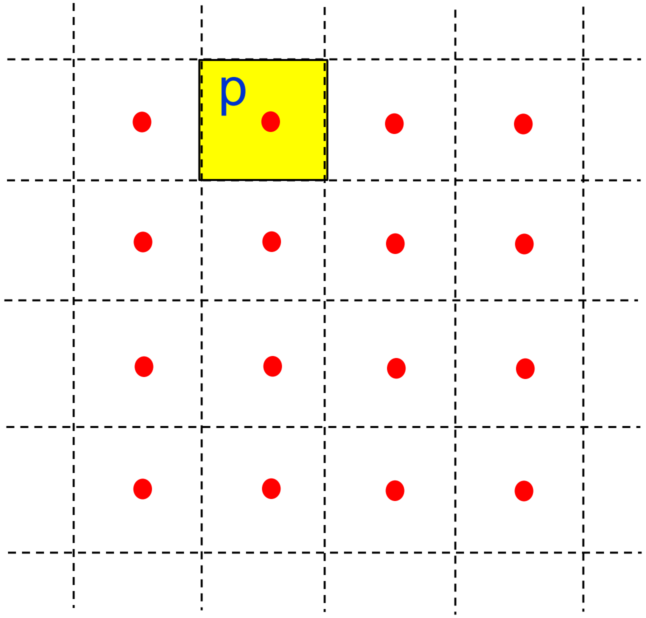
A backwards procedure is executed to eliminate p and all its ingoing transitions from the controller, until a controller is found which is non-blocking.

Integrated Algorithm for Problem 1

Successive iterations:

Repeat the procedure for all the target states.

The algorithm terminates when there are no more target states to be visited.



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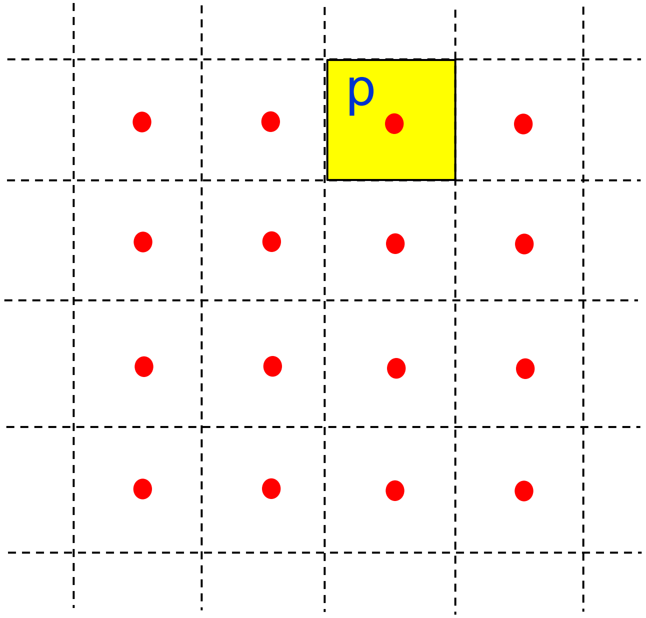

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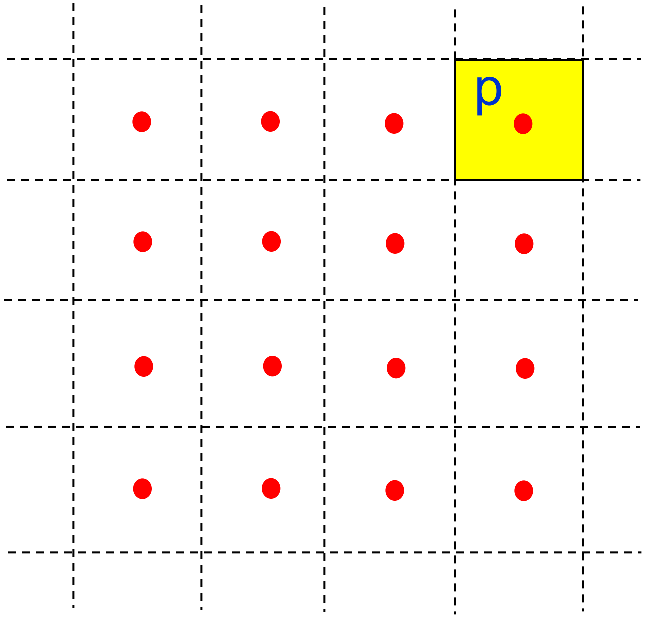


Integrated Algorithm for Problem 1

Properties

Let C** be the outcome of the integrated procedure:

1. The integrated algorithm terminates in a finite number of steps
2. C** and Nb(C*) are exactly bisimilar ⇒ C** solves Problem 1
3. C** is the minimal 0-bisimilar system of Nb(C*)
4. C** is accessible
5. space/time complexity of the integrated procedure is not larger than the one of the classical procedure

Integrated symbolic control design

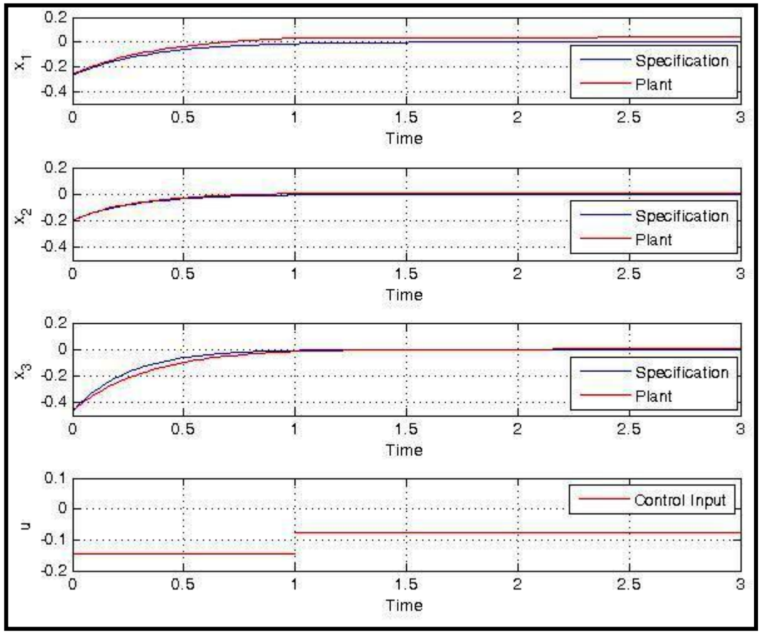
Example 1

Plant and Specification systems:

$$P : \begin{cases} \dot{x}_1 = -2x_1 + x_3^2 - u \\ \dot{x}_2 = 2x_1 - 7e^{x_2} + 7 \\ \dot{x}_3 = -3x_3 + \frac{3}{4}u^2, \end{cases}$$

$$Q : \begin{cases} \dot{x}_1 = -3x_1 + x_3^3 \\ \dot{x}_2 = x_1 - 5\sin x_2 \\ \dot{x}_3 = -x_2^2 - 4x_3. \end{cases}$$

Precision ε = 0.2

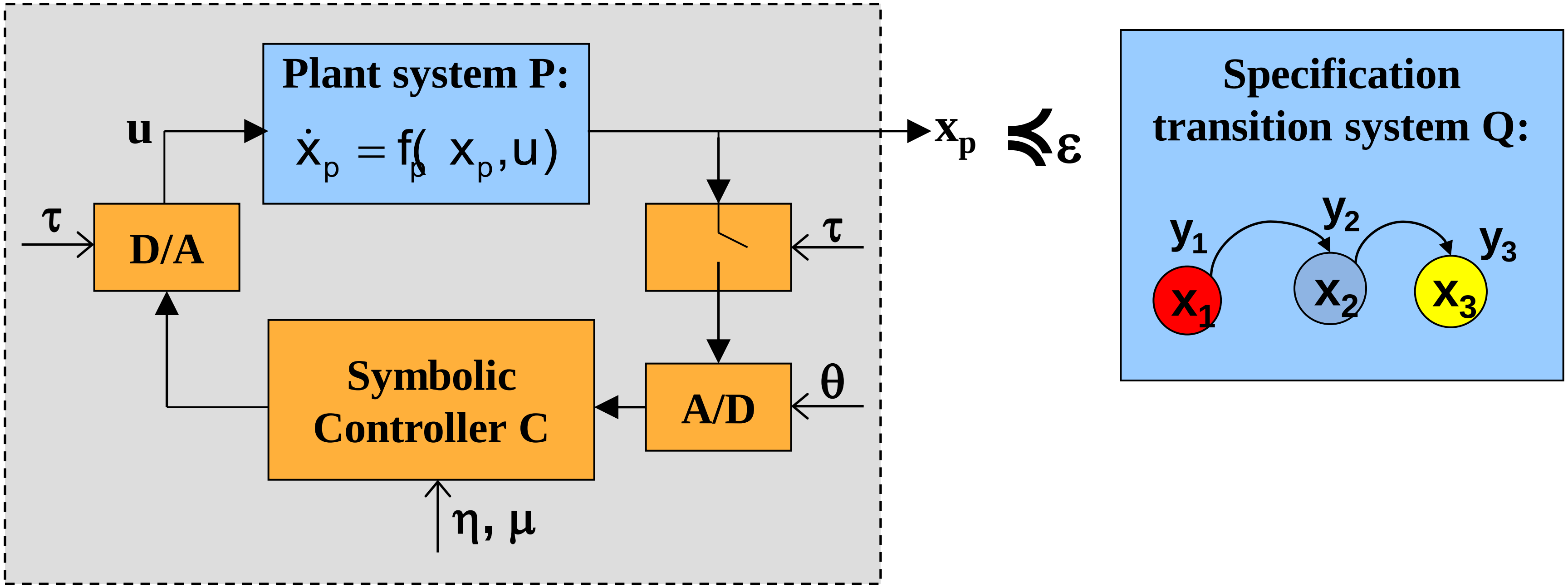


Comparison between Nb(C*) and C**	Nb(C*)	C**	Ratio
States	21,894	3,152	0.14
Transitions	12,652	3,152	$2.5 \cdot 10^{-3}$
Max memory occupation	93,347,397	10,400	$1.11 \cdot 10^{-4}$
Time	147,487	11,144	0.08

Design of symbolic controllers

Problem 2: Specifications given as deterministic transition systems [cf. Pola, Borri, Di Benedetto, IEEE TAC 12]

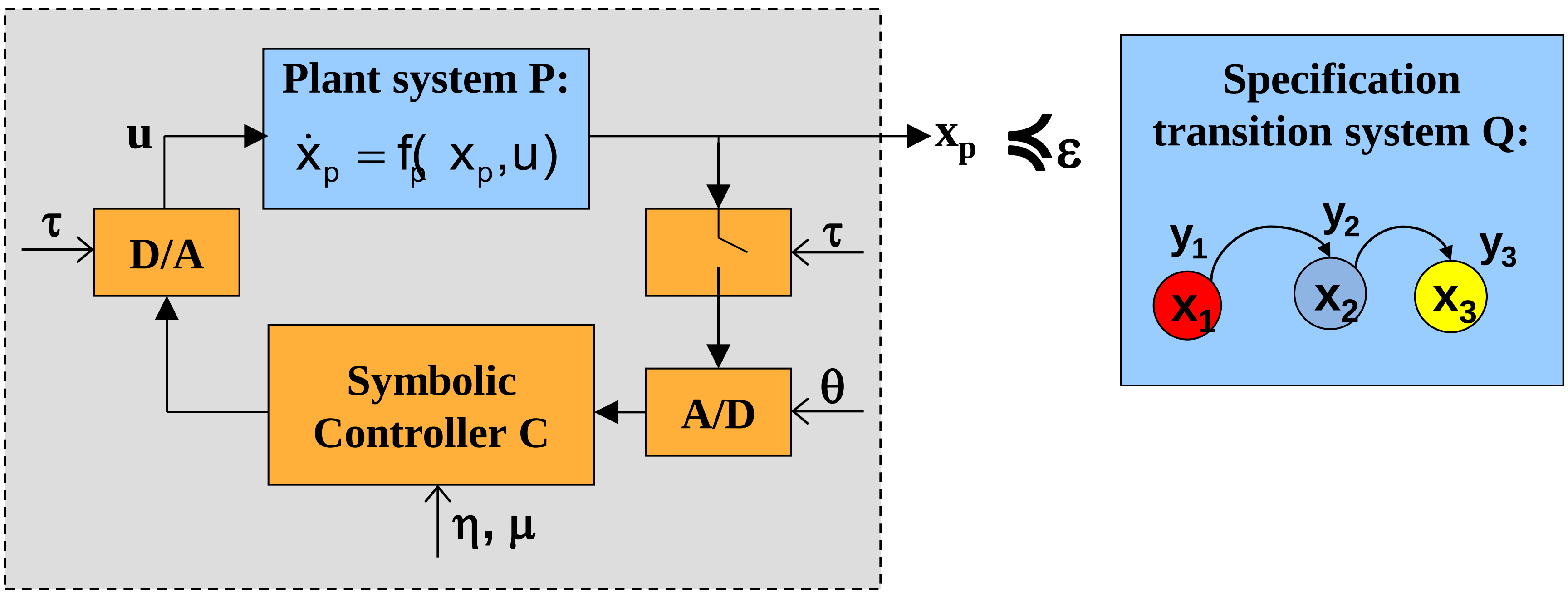
Given a plant P, a deterministic specification Q and a desired precision $\varepsilon > 0$, find a symbolic controller that implements Q up to the precision ε and that is non-blocking when interacting with P.



Design of symbolic controllers

Problem 2
Given a plant P, a deterministic specification Q and a desired precision $\varepsilon > 0$, find a symbolic controller C such that

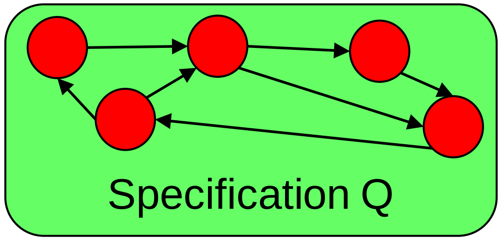
1. $T_\tau(P)\|_\theta C \preccurlyeq_\varepsilon Q$
2. $T_\tau(P)\|_\theta C$ is non-blocking



Solution of Problem 2

Synthesis through a three-step process:

1. Compute the symbolic model $T_{\tau,\eta,\mu}(P)$ of P
2. Compute the symbolic controller $C^* = T_{\tau,\eta,\mu}(P) ||_{\eta} Q$
3. Compute the non-blocking part Nb(C*) of C*



Plant P:
Continuous System

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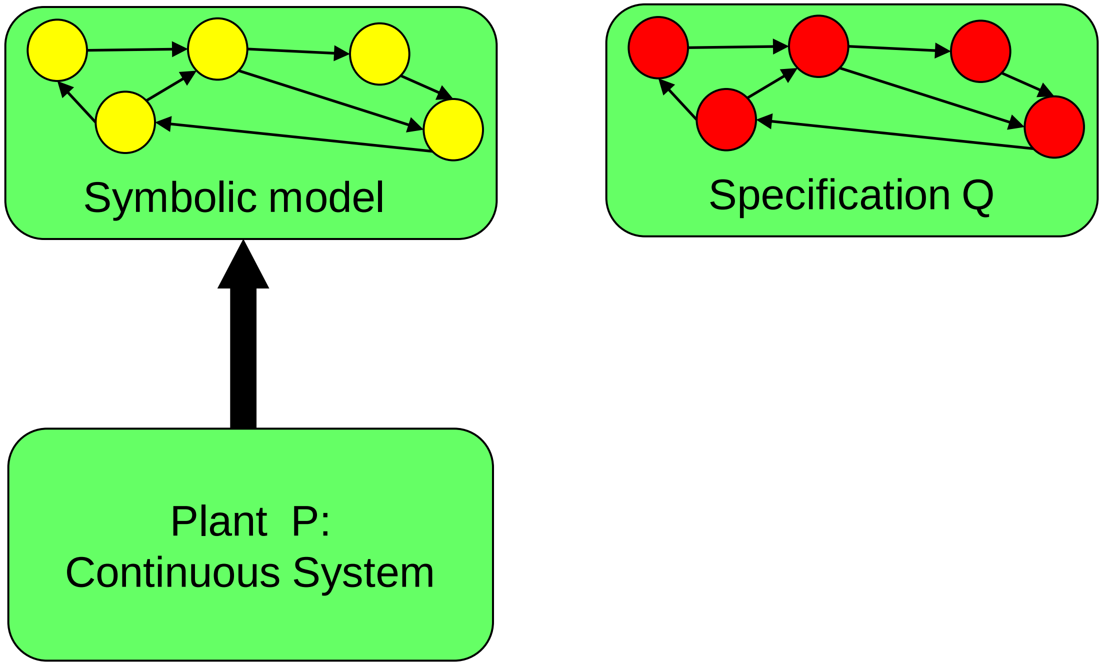

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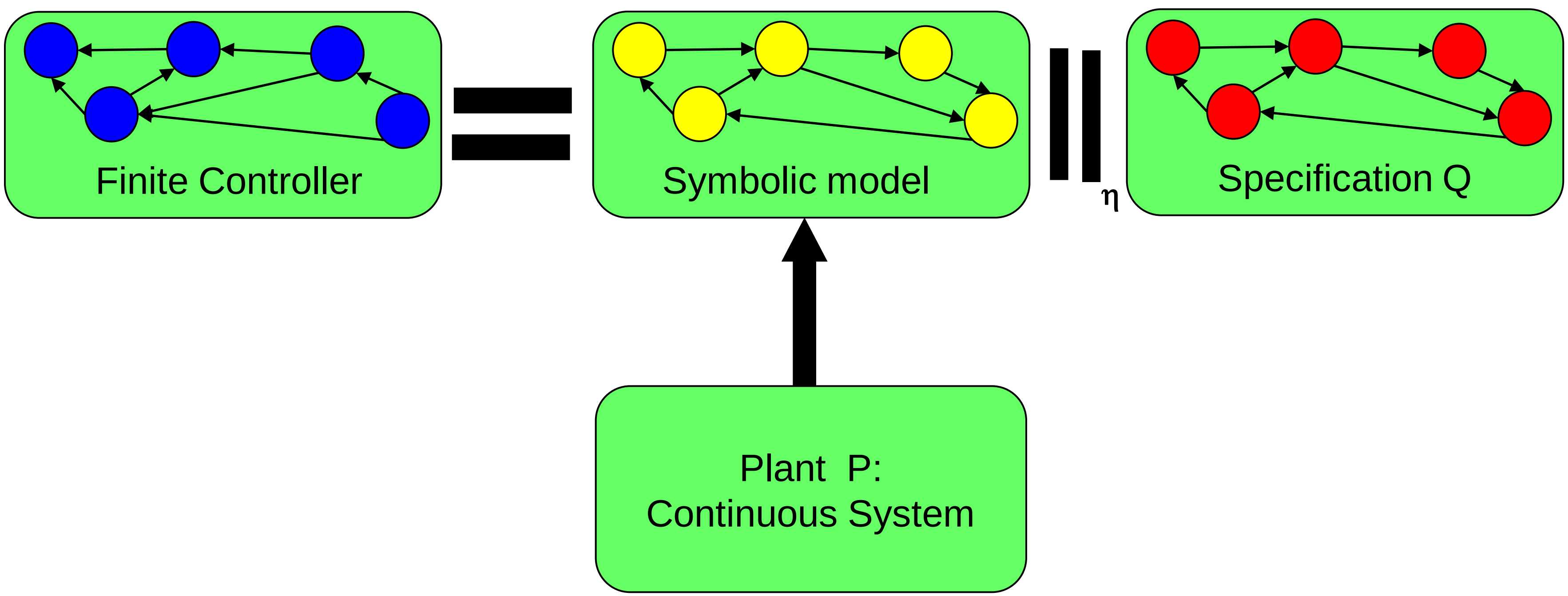

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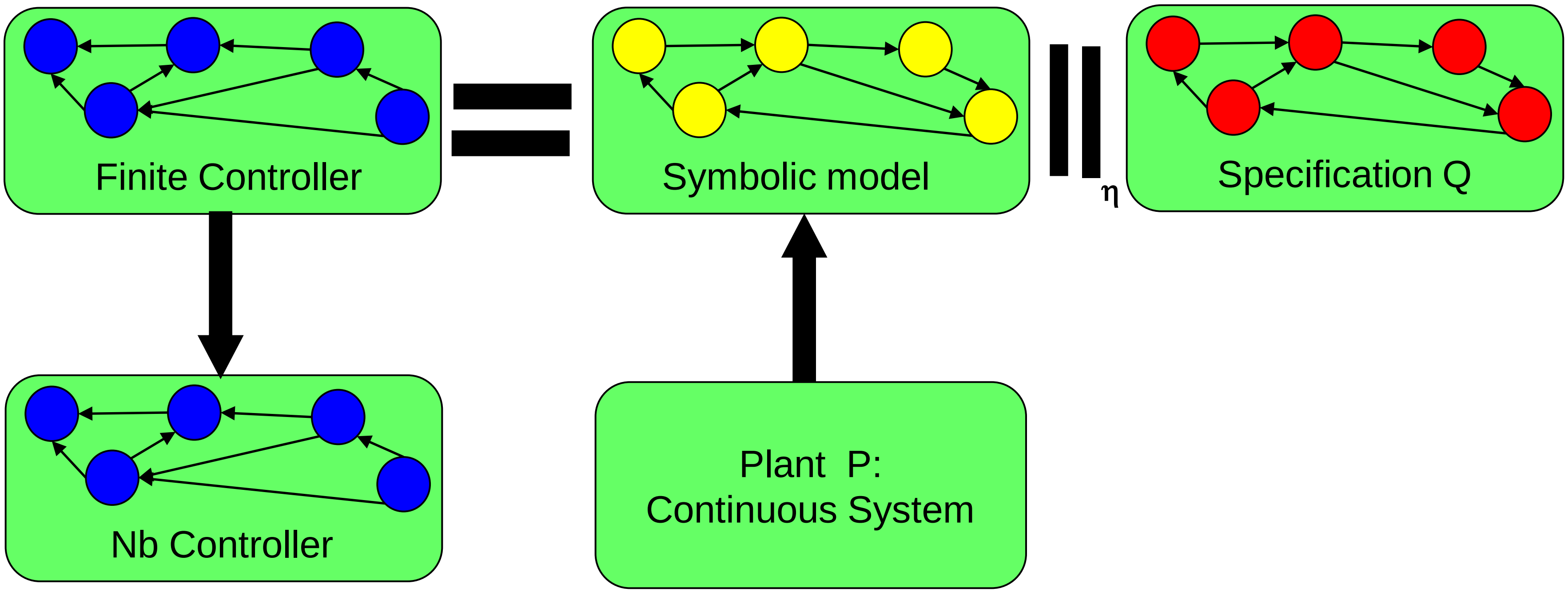

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Theorem Suppose that P is δ–ISS and choose parameters $\tau, \eta, \mu, \theta > 0$ satisfying:

$$\beta(\theta,\tau) + \gamma(\mu) + 2\eta \leq \theta + \eta \leq \varepsilon$$

The symbolic controller $Nb(C^*)$ solves Problem 2.

Design of symbolic controllers

Drawbacks

- It considers the whole sets of states of $T_{\tau,\eta,\mu}(P)$ and Q
- For any source state x and target state y, it includes all transitions $x \xrightarrow{u} y$ with any control input u by which state x reaches state y
- It first constructs $T_{\tau,\eta,\mu}(P)$ and Q, then C^*, to finally eliminate blocking states from C^*

To cope with space and time complexity, instead of computing separately

(1) Discrete abstraction $T_{\tau,\eta,\mu}(P)$ of P
(2) Symbolic controller $C^* = T_{\tau,\eta,\mu}(P)||_{\eta} Q$
(3) Non-blocking part $Nb(C)$ of C^*

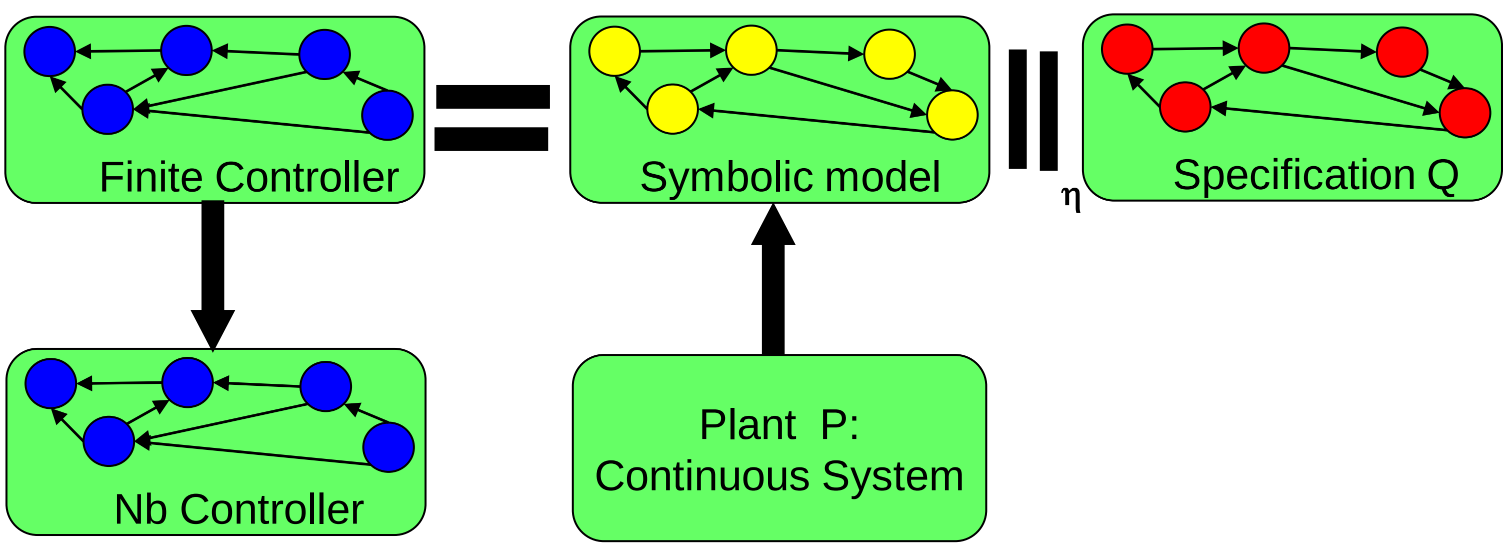
<u>Integrated Approach: Compute (1) + (2) + (3) at once!</u>

Space/time complexity analysis of the proposed algorithm formally quantifies the gain of the integrated approach

Integrated Algorithm for Problem 2

Basic ideas

1. It only considers the intersection of the accessible parts of P and Q
2. For any given source state x and target state y, it considers only one transition (x,u,y)
3. It eliminates blocking states as soon as show up



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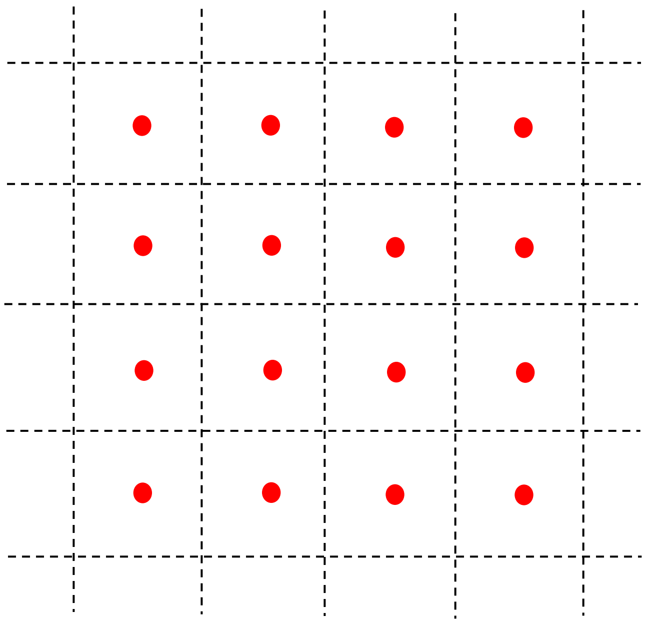

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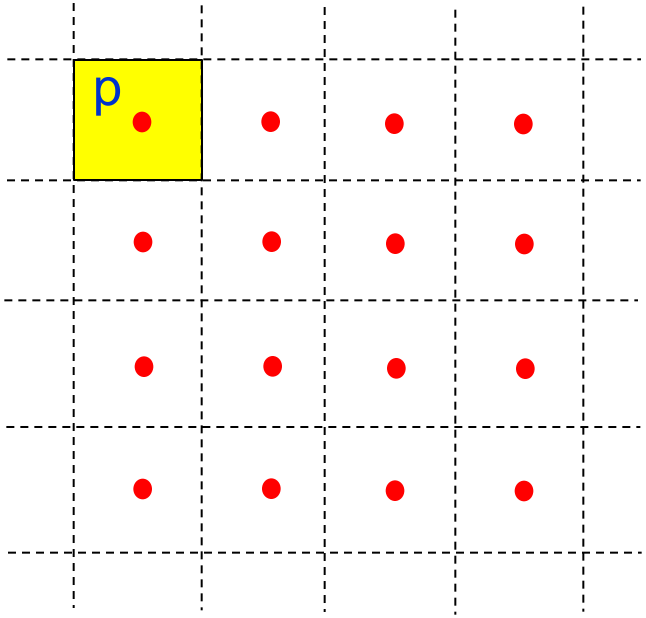

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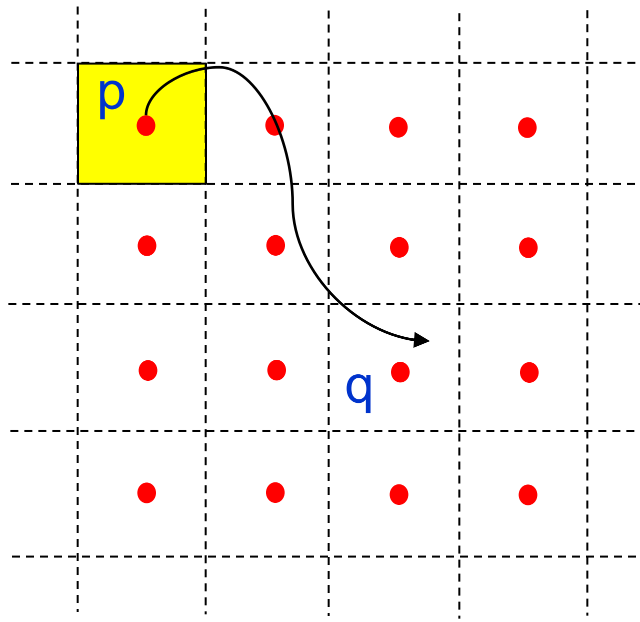

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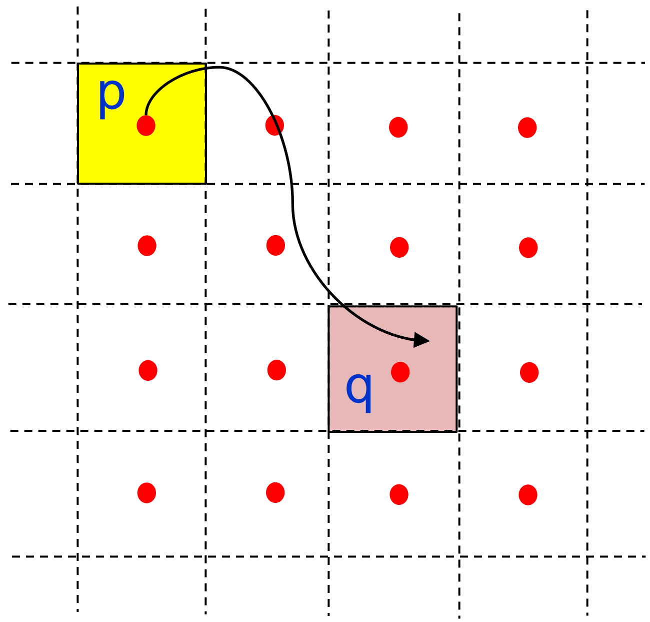

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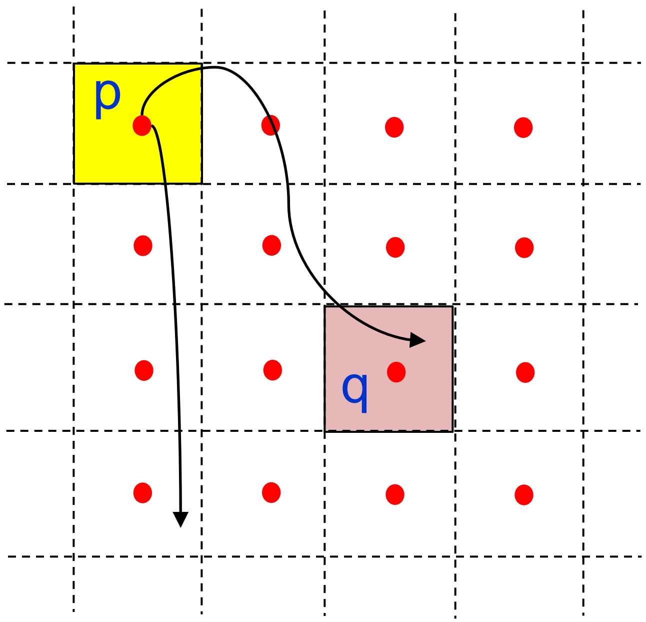

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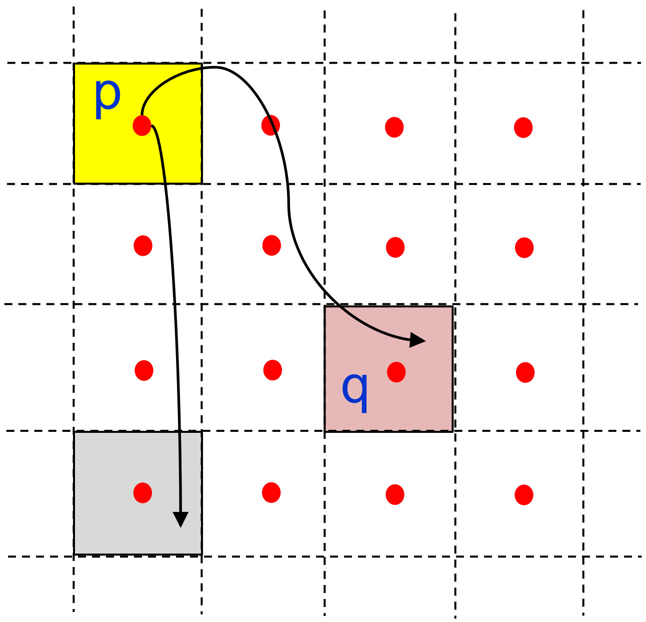

No matching! Try another input!

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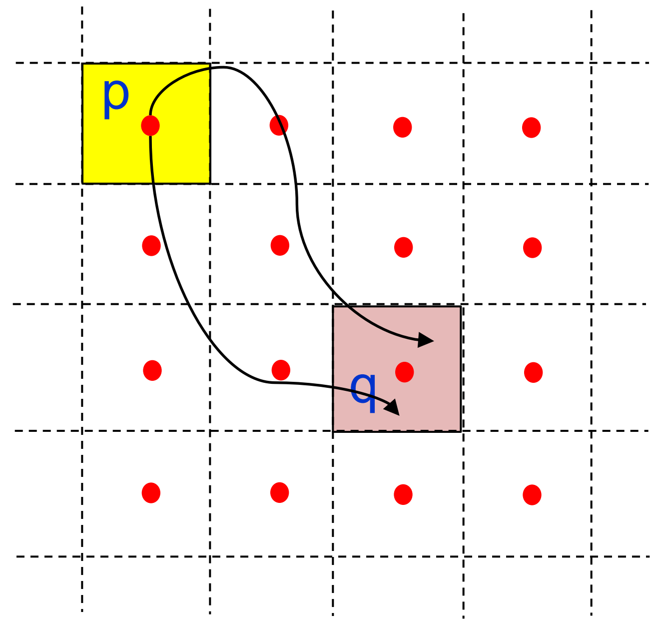

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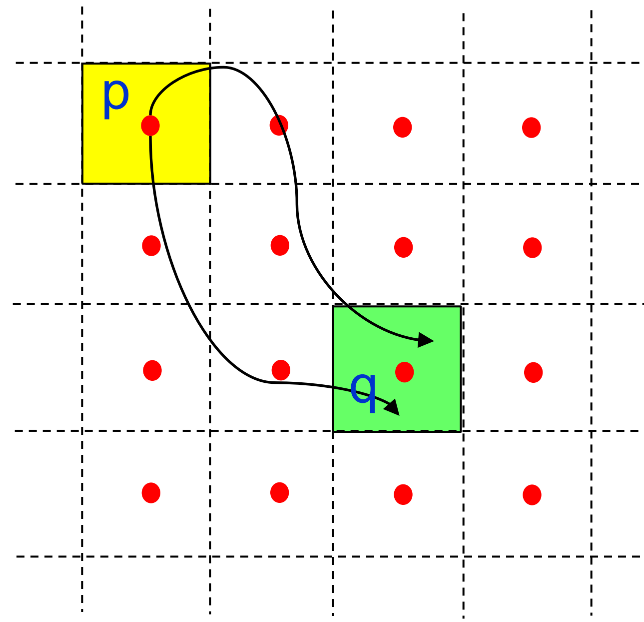

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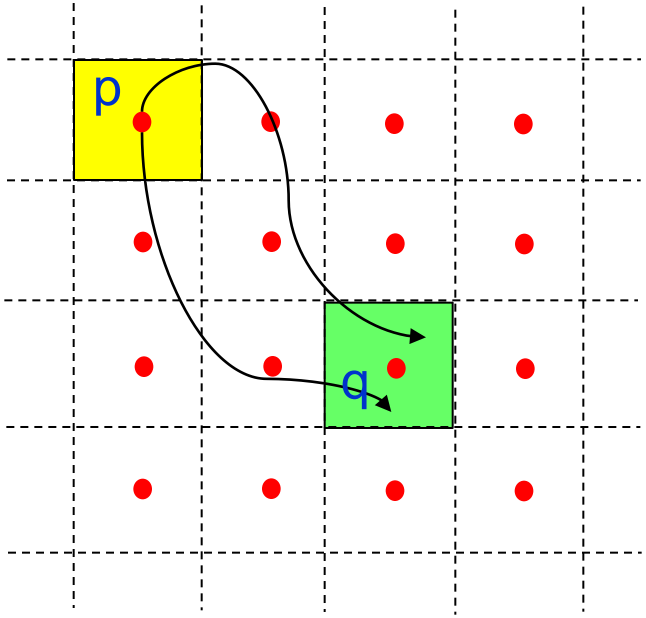

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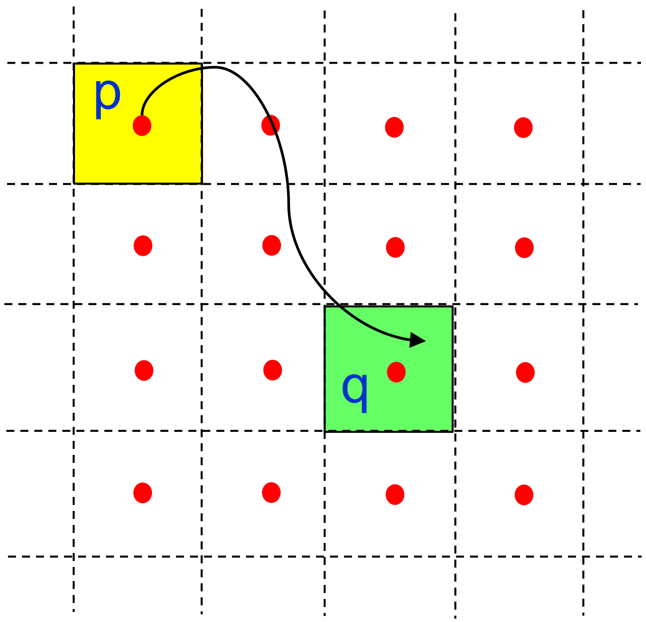

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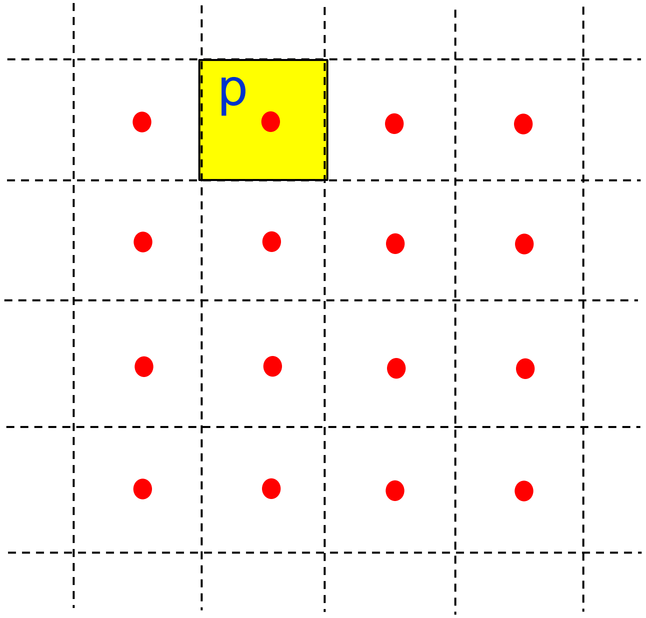
Successive iterations:

Repeat the procedure for all the target states.

The algorithm terminates when there are no more target states to be visited.



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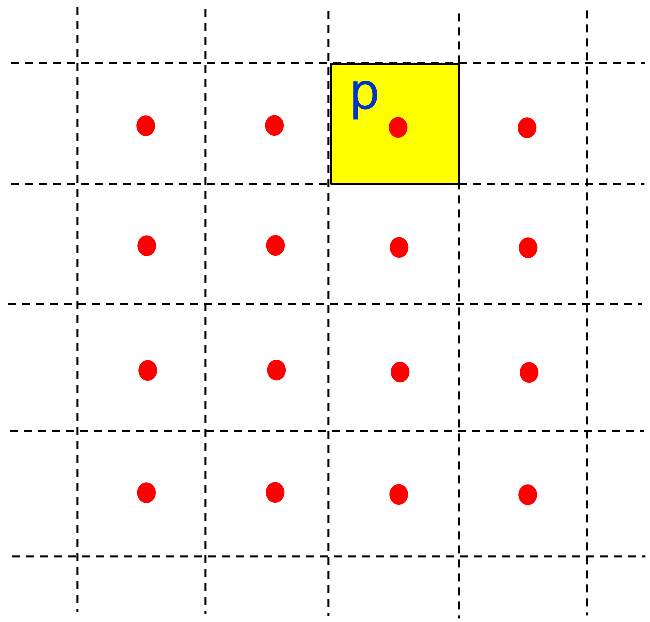

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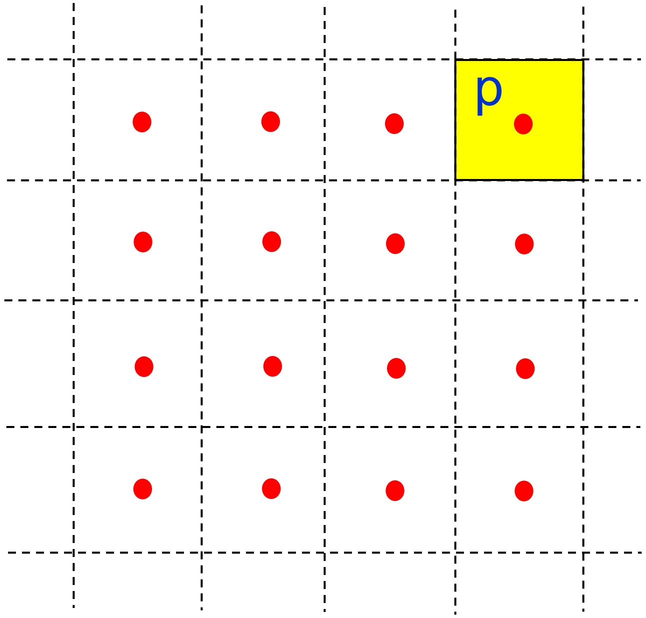

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Integrated symbolic control design

Example 2

Plant

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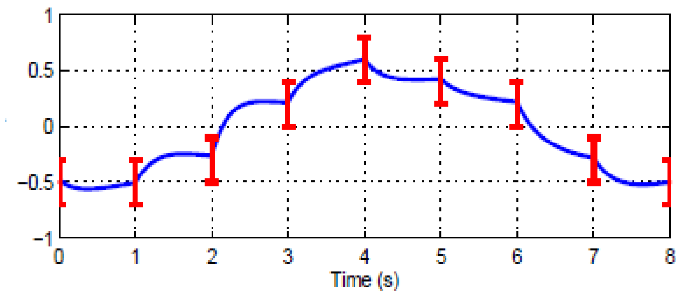
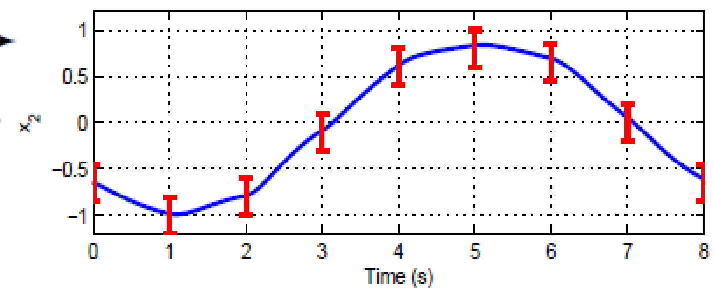


Specification

$$(-0.5, -0.65) \longrightarrow (-0.5, -1) \longrightarrow (-0.3, -0.8) \longrightarrow (0.2, -0.1) \longrightarrow (0.6, 0.6) \longrightarrow (0.4, 0.8) \longrightarrow (0.2, 0.65) \longrightarrow (-0.3, 0) \longrightarrow (-0.5, -0.65)$$



Precision ε = 0.2

Comparison between Nb(C*) and C**	Nb(C*)	C**	Ratio
Max memory occupation	2,759,580	48	$1.74 \cdot 10^{-5}$
Time	5,442	13	0.002